

# Generalized $q$ -Calkin-Wilf trees and $c$ -hyper $m$ -expansions of integers

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## Abstract

A hyperbinary expansion of a positive integer  $n$  is a partition of  $n$  into powers of 2 in which each part appears at most twice. In this paper, we consider a generalization of this concept and a certain statistic on the corresponding set of expansions of  $n$ . We then define  $q$ -generalized  $m$ -ary trees whose vertices are labeled by ratios of two consecutive terms within the sequence of distribution polynomials for the aforementioned statistic. When  $m = 2$ , we obtain a variant of a previously considered  $q$ -Calkin-Wilf tree.

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## 1 Introduction

The *Calkin-Wilf tree* (see, e.g., [1, 3]) is a binary tree having root  $\frac{1}{1}$  in which a vertex labeled  $\frac{a}{b}$  has two children, namely,  $\frac{a}{a+b}$  (the left child) and  $\frac{a+b}{b}$  (the right one). See Figure 1 below. Calkin and Wilf [3] have shown that each positive rational number appears exactly once in this tree, as a fraction in lowest terms. The Calkin-Wilf sequence is obtained by reading the tree line-by-line from left to right. It starts with

$$\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}, \frac{3}{1}, \frac{4}{1}, \frac{3}{5}, \frac{5}{2}, \frac{2}{5}, \frac{5}{3}, \frac{3}{4}, \frac{4}{1}, \dots,$$

and it was found by Newman (see Knuth [6]) that this sequence satisfies the somewhat unusual recurrence

$$x_{n+1} = \frac{1}{2 \lfloor x_n \rfloor + 1 - x_n}, \quad n \geq 1, \quad (1)$$

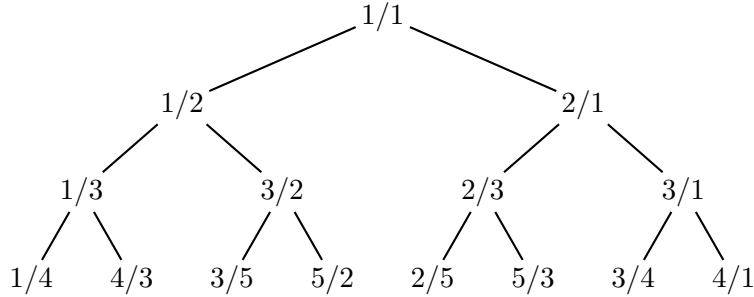


Figure 1: The first four levels of the Calkin-Wilf tree.

with initial condition  $x_1 = 1$ . This sequence was investigated as early as 1858 by Stern [12] (see also Reznick [9] and the references contained therein). Here, we will consider some related generalized trees that extend a certain aspect of the preceding sequence.

The diatomic sequence  $b_n$  is obtained by listing in order the numerators of the terms of the Calkin-Wilf sequence and starts with

$$1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, \dots$$

It is defined recursively by

$$b_{2n} = b_n, \quad b_{2n+1} = b_n + b_{n+1}, \quad n \geq 1,$$

with  $b_1 = 1$ , and has been an object of recent study (see, for example, [2, 11, 13] and the references contained therein). Various polynomial generalizations [4, 5] of the sequence  $b_n$  have been considered. For example, Klavžar et al. [5] defined the polynomials

$$B_{2n}(t) = tB_n(t), \quad B_{2n+1}(t) = B_n(t) + B_{n+1}(t), \quad n \geq 1, \quad (2)$$

with  $B_0(t) = 0$  and  $B_1(t) = 1$ , and Dilcher and Stolarsky [4] defined

$$F_{2n}(q) = F_n(q), \quad F_{2n+1}(q) = qF_n(q) + F_{n+1}(q), \quad n \geq 1, \quad (3)$$

with  $F_0(q) = F_1(q) = 1$ . Recently, Mansour [8] studied a  $q$ -analogue of the polynomials  $B_n(t)$  given by

$$B_{2n}(q, t) = tB_n(q, t), \quad B_{2n+1}(q, t) = qB_n(q, t) + B_{n+1}(q, t), \quad n \geq 1,$$

with the same initial conditions. Bates and Mansour [2] used the polynomials defined by (3) to define the  $q$ -analogue of the Calkin-Wilf tree, and found a statistic on the set of hyperbinary expansions of  $n$  for which  $F_{n+1}(q)$  is the distribution polynomial.

We will be considering an enumeration related to the following set of expansions of  $n$  into powers of a given positive integer  $m$ .

**Definition 1.1.** Fix  $m \geq 2$  and  $0 \leq c \leq m - 1$ . By a  $c$ -hyper  $m$ -expansion of a positive integer  $n$ , we mean a partition of  $n$  into powers of  $m$  in which a given power can appear exactly  $j$  times, where  $j \in \{0, 1, \dots, m - 1, m + c\}$ .

Note that the  $m = 2, c = 0$  case of the preceding definition corresponds to the hyperbinary expansions of  $n$ . We consider the following related sequence of polynomials.

**Definition 1.2.** Given  $m \geq 2$  and  $0 \leq c \leq m - 1$ , define the sequence of polynomials  $f_{m,c}(d; q)$  for  $d \geq 0$  by

$$\begin{aligned} f_{m,c}(mn + j; q) &= f_{m,c}(n; q), & j = 0, 1, \dots, c - 1, c + 1, \dots, m - 1, \\ f_{m,c}(mn + c; q) &= f_{m,c}(n; q) + qf_{m,c}(n - 1; q), \end{aligned} \quad (4)$$

with  $f_{m,c}(0; q) = 1$  and  $f_{m,c}(d; q) = 0$  for  $d < 0$ .

Note that the  $f_{m,c}(n; q)$  provide a generalization of the sequence defined by (3) in that  $f_{2,0}(n; q) = F_{n+1}(q)$  for all  $n \geq 0$ .

The presentation of this paper is as follows. In the next section, we provide combinatorial interpretations for the polynomials  $f_{m,c}(n; q)$  in terms of  $c$ -hyper  $m$ -expansions of  $n$ . In the third section, we describe  $m$ -ary trees whose vertices are labeled by ratios of consecutive terms of the sequence  $f_{m,c}(n; q)$ . Different trees are needed depending on whether  $c = m - 1$ ,  $c = 0$  or  $1 \leq c \leq m - 2$ . When  $m = 2$ , one obtains a variant of the  $q$ -Calkin-Wilf tree considered in [2]. In the case  $c = m - 1$ , the rational functions labeling the vertices of certain branches of the tree may be expressed in terms of Chebyshev polynomials of the second kind. Furthermore, for all  $c$ , it is shown that each rational number in the interval  $(0, 1]$  appears at least once in the corresponding  $m$ -ary tree when  $q = 1$ .

## 2 $(q, c)$ -hyper $m$ -expansions of the number $n$

We first define the concept of a  $(q, c)$ -hyper  $m$ -expansion of the number  $n$ .

**Definition 2.1.** Fix  $m \geq 2$  and  $0 \leq c \leq m - 1$ . We denote the set of all  $c$ -hyper  $m$ -expansions of a positive integer  $n$  by  $\mathbb{H}_{m,c,n}$  and the number of powers that are used exactly  $m + c$  times in the hyper  $m$ -expansion  $x \in \mathbb{H}_{m,c,n}$  by  $h_{m,c,n}(x)$ . The  $(q, c)$ -hyper  $m$ -expansion of  $x$  is defined as  $q^{h_{m,c,n}(x)}$ .

**Definition 2.2.** Let  $g_{m,c}(n; q)$  be the polynomial consisting of the sum of  $(q, c)$ -hyper  $m$ -expansions of  $n$ , with  $g_{m,c}(0; q) = 1$  and  $g_{m,c}(r; q) = 0$  if  $r < 0$ .

For example, the 2-hyper 3-expansions of 47 are  $27 + 9 + 9 + 1 + 1$ ,  $27 + 9 + 3 + 3 + 1 + 1 + 1 + 1 + 1$ ,  $9 + 9 + 9 + 9 + 9 + 9 + 1 + 1$  and  $27 + 3 + 3 + 3 + 3 + 3 + 1 + 1 + 1 + 1 + 1$ . Thus, the  $(q, 2)$ -hyper 3-expansions of 47 are  $q^0$ ,  $q^1$ ,  $q^1$  and  $q^2$  and, accordingly,  $g_{3,2}(47; q) = 1 + 2q + q^2$ .

**Theorem 2.3.** For all  $n \geq 0$ ,  $g_{m,c}(n; q) = f_{m,c}(n; q)$ .

*Proof.* We proceed by induction on  $n$ . Since  $g_{m,c}(0; q) = 1 = f_{m,c}(0; q)$ , the claim holds for  $n = 0$ . Assume that the claim holds for  $0, 1, \dots, n-1$  and let us prove it for  $n$ . By the induction hypothesis and (4), we have that

- if  $n = mr + j$  with  $j \in \{0, 1, \dots, c-1, c+1, \dots, m-1\}$ , then

$$\begin{aligned} g_{m,c}(n; q) &= g_{m,c}(mr + j; q) = g_{m,c}(r; q) \\ &= f_{m,c}(r; q) = f_{m,c}(mr + j; q) = f_{m,c}(n; q), \end{aligned}$$

- if  $n = mr + c$ , then

$$\begin{aligned} g_{m,c}(n; q) &= g_{m,c}(mr + c; q) = g_{m,c}(r; q) + qg_{m,c}(r-1; q) \\ &= f_{m,c}(r; q) + qf_{m,c}(r-1; q) = f_{m,c}(mr + c; q) = f_m(n; q). \end{aligned}$$

This completes the induction.  $\square$

### 3 $(q, c)$ -Calkin-Wilf trees of order $m$

In this section, we define  $m$ -ary trees whose vertices are labeled with certain rational functions of  $q$ , in particular, by ratios of consecutive terms of the  $g_{m,c}(n; q)$  sequence.

#### 3.1 Case $c = m - 1$

In this subsection, we construct an  $m$ -ary tree whose vertices are labeled by the ratios of certain terms within the  $g_{m,c}(n; q)$  sequence in the case when  $c = m - 1$  and consider some of its properties.

**Definition 3.1.** *If  $m \geq 3$ , then the  $(q, m-1)$ -Calkin-Wilf tree of order  $m$  is an  $m$ -ary tree with root  $\frac{1}{1}$ . A vertex labeled  $\frac{a}{b}$  is a parent of  $m$  children defined, from left to right, as follows. Each of the first  $m-2$  children is  $\frac{1}{1+q}$ , with the  $(m-1)$ -st child given by  $\frac{b}{b+qa}$ . To define the  $m$ -th child, suppose  $\frac{a_j}{b_j}$  is the  $(m-1)$ -st child of  $\frac{a_{j+1}}{b_{j+1}}$  for  $j = 1, 2, \dots, s-1$ , where  $\frac{a_1}{b_1} = \frac{a}{b}$  and  $s \geq 1$  is maximal. Then the  $m$ -th child of  $\frac{a}{b}$  is given by  $\frac{1}{1+qp \prod_{j=1}^s \frac{b_j}{a_j}}$ , where  $p = \frac{a_{s+1}}{b_{s+1}}$  if  $\frac{a_s}{b_s}$  is the  $(m-2)$ -nd child of  $\frac{a_{s+1}}{b_{s+1}}$  and  $p = 1$  otherwise.*

The following figure illustrates the  $(q, 2)$ -Calkin-Wilf tree of order 3 when  $q = 1$ .



$n_s = mn' + j$  for some  $j \in \{1, 2, \dots, m-2, m\}$  or  $n_s = 0$ . By the definitions, we have

$$\begin{aligned}
& \frac{g_{m,m-1}(m(mn+m)+m-2;q)}{g_{m,m-1}(m(mn+m)+m-1;q)} \\
&= \frac{g_{m,m-1}(mn+m;q)}{g_{m,m-1}(mn+m;q) + qg_{m,m-1}(mn+m-1;q)} \\
&= \frac{g_{m,m-1}(n+1;q)}{g_{m,m-1}(n+1;q) + q(g_{m,m-1}(n;q) + qg_{m,m-1}(n-1;q))} \\
&= \frac{1}{1 + q \frac{g_{m,m-1}(n;q) + qg_{m,m-1}(n-1;q)}{g_{m,m-1}(n;q)} \frac{g_{m,m-1}(n;q)}{g_{m,m-1}(n+1;q)}},
\end{aligned}$$

with

$$\begin{aligned}
\frac{g_{m,m-1}(n_1;q)}{g_{m,m-1}(n_1+1;q)} &= \frac{g_{m,m-1}(mn_2+m-1;q)}{g_{m,m-1}(mn_2+m;q)} \\
&= \frac{g_{m,m-1}(n_2;q) + qg_{m,m-1}(n_2-1;q)}{g_{m,m-1}(n_2+1;q)} \\
&= \frac{g_{m,m-1}(n_2;q) + qg_{m,m-1}(n_2-1;q)}{g_{m,m-1}(n_2;q)} \frac{g_{m,m-1}(n_2;q)}{g_{m,m-1}(n_2+1;q)} \\
&= \dots \\
&= \frac{g_{m,m-1}(n_s;q)}{g_{m,m-1}(n_s+1;q)} \prod_{j=2}^s \frac{g_{m,m-1}(n_j;q) + qg_{m,m-1}(n_j-1;q)}{g_{m,m-1}(n_j;q)}.
\end{aligned}$$

Note that if  $j \in [m-3]$  or if  $n_s = 0$ , then

$$\frac{g_{m,m-1}(n_s;q)}{g_{m,m-1}(n_s+1;q)} = \frac{g_{m,m-1}(n';q)}{g_{m,m-1}(n';q)} = 1.$$

If  $j = m$ , then

$$\frac{g_{m,m-1}(n_s;q)}{g_{m,m-1}(n_s+1;q)} = \frac{g_{m,m-1}(n'+1;q)}{g_{m,m-1}(n'+1;q)} = 1.$$

If  $j = m-2$ , then

$$\frac{g_{m,m-1}(n_s;q)}{g_{m,m-1}(n_s+1;q)} = \frac{g_{m,m-1}(mn'+m-2;q)}{g_{m,m-1}(mn'+m-1;q)} = \ell_m(n';q).$$

Hence,

$$\begin{aligned}
\frac{g_{m,m-1}(n_1;q)}{g_{m,m-1}(n_1+1;q)} &= \frac{g_{m,m-1}(n_s;q)}{g_{m,m-1}(n_s+1;q)} \prod_{j=2}^s \frac{g_{m,m-1}(n_j;q) + qg_{m,m-1}(n_j-1;q)}{g_{m,m-1}(n_j;q)} \\
&= p \prod_{j=2}^s \frac{g_{m,m-1}(n_j;q) + qg_{m,m-1}(n_j-1;q)}{g_{m,m-1}(n_j;q)},
\end{aligned}$$

where  $p$  is as defined above. Thus, by the induction hypothesis and the definitions, we have

$$\begin{aligned}
\frac{g_{m,m-1}(m(mn+m)+m-2;q)}{g_{m,m-1}(m(mn+m)+m-1;q)} &= \frac{1}{1+qp \prod_{j=1}^s \frac{g_{m,m-1}(n_j;q)+qg_{m,m-1}(n_j-1;q)}{g_{m,m-1}(n_j;q)}} \\
&= \frac{1}{1+qp \prod_{j=1}^s \frac{g_{m,m-1}(mn_j+m-1;q)}{g_{m,m-1}(mn_j+m-2;q)}} \\
&= \frac{1}{1+qp \prod_{j=1}^s \frac{1}{\ell_m(n_j;q)}} \\
&= \ell_m(mn+m;q).
\end{aligned}$$

Hence, the children of  $\ell_m(n;q) = \frac{g_{m,m-1}(mn+m-2;q)}{g_{m,m-1}(mn+m-1;q)}$  in  $\mathcal{T}_m$  are

$$\left\{ \ell_m(mn+j;q) = \frac{g_{m,m-1}(m(mn+j)+m-2;q)}{g_{m,m-1}(m(mn+j)+m-1;q)} \right\}_{j=1}^m,$$

which completes the induction.  $\square$

*Remark:* The result of the previous theorem also holds when  $m = 2$ , provided that the  $p$  factor in Definition 3.1 is adjusted in the case  $m = 2$  as follows: let  $p = \ell_2(n'+1;q)$  if  $n_s = 2n'+2$ , where  $n_s$  and  $n'$  are as defined in the prior proof, and  $p = 1$  otherwise.

**Definition 3.3.** Let  $v$  be any vertex of the  $(q, m-1)$ -Calkin-Wilf tree of order  $m$  and let  $j \in [m]$ . The set of all vertices  $v_1 = v, v_2, v_3, \dots$  such that  $v_i$  is the  $j$ -th child of  $v_{i-1}$  for  $i \geq 2$  will be denoted by  $B_{v,j}$  and will be called the  $j$ -th branch of  $v$ .

For example, the  $m$ -th branch of the root  $1/1$  is given by  $B_{1/1,m} = \{1/1, 1/(1+q), 1/(1+q+q^2), 1/(1+q+q^2+q^3), \dots\}$  for  $m \geq 3$ .

In order to state our next result, we recall the Chebyshev polynomials of the second kind (see [10]) defined by the recurrence

$$U_n(t) = 2tU_{n-1}(t) - U_{n-2}(t), \quad n \geq 2, \quad (5)$$

with  $U_0(t) = 1$  and  $U_1(t) = 2t$ .

**Theorem 3.4.** The  $(m-1)$ -st branch of the root in the  $(q, m-1)$ -Calkin-Wilf tree of order  $m \geq 2$  is given by

$$B_{1/1,m-1} = \left\{ \frac{U_j\left(\frac{1}{2\sqrt{-q}}\right)}{\sqrt{-q}U_{j+1}\left(\frac{1}{2\sqrt{-q}}\right)} \right\}_{j \geq 0}.$$

If  $m \geq 3$ , then the  $j$ -th branch of the root is given by  $B_{1/1,j} = \{1/1, 1/(1+q), 1/(1+q), 1/(1+q), \dots\}$  for  $j \in [m-2]$ , and the  $m$ -th branch is given by  $B_{1/1,m} = \{1/1, 1/(1+q), 1/(1+q+q^2), 1/(1+q+q^2+q^3), \dots\}$ .

*Proof.* Let  $x_n = \frac{g_{m,m-1}(mn+m-2;q)}{g_{m,m-1}(mn+m-1;q)}$  for  $n \geq 0$ . By induction, one can show that the  $(m-1)$ -st branch is given by  $B_{1/1,m-1} = \{x_{m^j-1}\}_{j \geq 0}$ . By (4), we have

$$\begin{aligned} x_{m^j-1} &= \frac{g_{m,m-1}(m(m^j-1)+m-2;q)}{g_{m,m-1}(m(m^j-1)+m-1;q)} \\ &= \frac{g_{m,m-1}(m^j-1;q)}{g_{m,m-1}(m^j-1;q) + qg_{m,m-1}(m^j-2;q)} \\ &= \frac{g_{m,m-1}(m(m^{j-1}-1)+m-1;q)}{g_{m,m-1}(m(m^{j-1}-1)+m-1;q) + qg_{m,m-1}(m(m^{j-1}-1)+m-2;q)} \\ &= \frac{1}{1 + qx_{m^{j-1}-1}}, \quad j \geq 1, \end{aligned}$$

with  $x_0 = 1$ . By induction on  $j$  and (5), we obtain

$$x_{m^j-1} = \frac{U_j\left(\frac{1}{2\sqrt{-q}}\right)}{\sqrt{-q}U_{j+1}\left(\frac{1}{2\sqrt{-q}}\right)}, \quad j \geq 0.$$

The second statement follows from the definitions.  $\square$

Similarly, one can show the following result.

**Theorem 3.5.** *Let  $v = \frac{U_j\left(\frac{1}{2\sqrt{-q}}\right)}{\sqrt{-q}U_{j+1}\left(\frac{1}{2\sqrt{-q}}\right)}$ . Then the  $(m-1)$ -st branch of  $v$  in the  $(q, m-1)$ -Calkin-Wilf tree of order  $m \geq 2$  is given by*

$$B_{v,m-1} = \left\{ \frac{U_i\left(\frac{1}{2\sqrt{-q}}\right)}{\sqrt{-q}U_{i+1}\left(\frac{1}{2\sqrt{-q}}\right)} \right\}_{i \geq j}.$$

*If  $m \geq 3$ , then the  $j$ -th branch of  $v$  is given by  $B_{v,j} = \{v, 1/(1+q), 1/(1+q), 1/(1+q), \dots\}$  for  $j \in [m-2]$ , and the  $m$ -th branch is given by  $B_{v,m} = \{v, 1/(1+qt), 1/(1+q+q^2t), 1/(1+q+q^2+q^3t), \dots\}$ , where  $t = \sqrt{-q}^{j+1}U_{j+1}\left(\frac{1}{2\sqrt{-q}}\right)$ .*

We conclude the case  $c = m-1$  with the following result when  $q = 1$  concerning the tree  $\mathcal{T}_m$ .

**Theorem 3.6.** *When  $q = 1$ , each positive rational number less than or equal one appears at least once in the  $(q, m-1)$ -Calkin-Wilf tree of order  $m$  for all  $m \geq 3$ .*

*Proof.* We will show that all rational numbers  $\frac{a}{b}$ , where  $0 < \frac{a}{b} \leq 1$ , belong to  $\mathcal{T}_m$  when  $m \geq 3$  and  $q = 1$ . First note that all fractions of the form  $\frac{1}{b}$  belong to  $\mathcal{T}_m$ , upon considering the  $m$ -th branch of the root.

So suppose  $0 < \frac{a}{b} < 1$  is a rational number (in lowest terms) such that  $a > 1$ . Let  $\frac{1}{x}$  be an element of the  $m$ -th branch of the root, where  $x > 1$  is to be determined. Let  $v = \frac{1}{2}$  be the  $(m-2)$ -nd child of  $\frac{1}{x}$ . Consider the  $(m-1)$ -st branch of  $v$ , the sequence of which we will denote by  $v_1 = v, v_2, v_3, \dots$ . Let  $v_i = \frac{a_i}{b_i}$  in reduced



form for  $i \geq 1$ . It can be shown by induction that  $a_i = f_i$  and  $b_i = f_{i+1}$ , where  $f_i$  denotes the Fibonacci sequence defined by  $f_i = f_{i-1} + f_{i-2}$  for  $i \geq 2$  with  $f_0 = f_1 = 1$ . If  $t \geq 1$ , then  $\prod_{i=1}^t \frac{b_i}{a_i} = f_{t+1}$ , which implies that the  $m$ -th child of  $v_t$  is  $\frac{1}{1 + \frac{f_{t+1}}{x}}$ , by the definitions.

Recall the well known fact (see, e.g., [7, p. 73-74]) that given any positive integer  $j$ , there exists some  $k$  such that  $j$  divides  $f_k$ . Choosing  $t$  so that  $b - a$  divides  $f_{t+1}$ , i.e.,  $f_{t+1} = u(b - a)$  for some  $u \geq 1$ , and then letting  $x = ua$ , implies that the  $m$ -th child of  $v_t$  is given by

$$\frac{1}{1 + \frac{f_{t+1}}{x}} = \frac{1}{1 + \frac{u(b-a)}{ua}} = \frac{a}{b}.$$

Thus, we have  $\frac{a}{b} \in \mathcal{T}_m$ , which completes the proof.  $\square$

### 3.2 Case $c = 0$

A comparable tree may be constructed in the case when  $c = 0$ .

**Definition 3.7.** *The  $(q, 0)$ -Calkin-Wilf tree of order  $m$  is an  $m$ -ary tree with root  $\frac{1}{1}$ . A vertex labeled  $\frac{a}{b}$  is a parent to  $m$  children defined, from left to right, as follows. Each of the first  $m - 2$  children is  $\frac{1}{1+q}$ , with the  $(m - 1)$ -st child given by  $\frac{a}{b+qa}$ . To define the  $m$ -th child, suppose  $\frac{a_j}{b_j}$  is the  $m$ -th child of  $\frac{a_{j+1}}{b_{j+1}}$  for  $j = 1, 2, \dots, s - 1$ , where  $\frac{a_1}{b_1} = \frac{a}{b}$  and  $s \geq 1$  is maximal. Then the  $m$ -th child of  $\frac{a}{b}$  is given by*

$$\frac{b}{qb + \frac{a}{1+q+\dots+q^{s-2}+q^{s-1}r}},$$

where  $r = \frac{a_{s+1}}{b_{s+1}}$  if  $\frac{a_s}{b_s}$  is the  $(m - 1)$ -st child of  $\frac{a_{s+1}}{b_{s+1}}$  and  $r = 1$  otherwise.

One can describe an  $m$ -ary tree in analogy to the case  $c = m - 1$  above whose vertices are labeled by rational functions of the form  $\frac{g_{m,0}(mn+m-1;q)}{g_{m,0}(mn+m;q)}$ .

**Theorem 3.8.** *Let  $m \geq 2$  and let the concatenation of successive levels of the  $(q, 0)$ -Calkin-Wilf tree of order  $m$  form a sequence  $\{\ell_m(n; q)\}_{n \geq 0}$ . Then*

$$\ell_m(n; q) = \frac{g_{m,0}(mn + m - 1; q)}{g_{m,0}(mn + m; q)},$$

for all  $n \geq 0$ .

*Proof.* Let  $\mathcal{T}_m$  be the  $(q, 0)$ -Calkin-Wilf tree of order  $m$ . We proceed by induction on  $n$ . Since  $\ell_m(0, q) = \frac{g_{m,0}(m-1;q)}{g_{m,0}(m;q)} = \frac{1}{1}$ , the claim holds for  $n = 0$ . Assume that the claim holds for  $\ell_m(0; q), \ell_m(1; q), \dots, \ell_m(n; q)$  of  $\mathcal{T}_m$  and let us prove it for the children of  $\ell_m(n; q)$ , which are  $\ell_m(mn + 1; q), \ell_m(mn + 2; q), \dots, \ell_m(mn + m; q)$ .

By the induction hypothesis and Definition 3.7, we have

$$\begin{aligned}
\frac{g_{m,0}(m(mn+j)+m-1;q)}{g_{m,0}(m(mn+j)+m;q)} &= \frac{g_{m,0}(mn+j;q)}{g_{m,0}(mn+j+1;q) + qg_{m,0}(mn+j;q)} \\
&= \frac{g_{m,0}(n;q)}{g_{m,0}(n;q) + qg_{m,0}(n;q)} \\
&= \frac{1}{1+q} = \ell_m(mn+j;q),
\end{aligned}$$

for all  $j \in [m-2]$ . Thus, the claim holds for  $\ell_m(mn+j;q)$  when  $j \in [m-2]$ . We also have

$$\begin{aligned}
\frac{g_{m,0}(m(mn+m-1)+m-1;q)}{g_{m,0}(m(mn+m-1)+m;q)} &= \frac{g_{m,0}(mn+m-1;q)}{g_{m,0}(mn+m;q) + qg_{m,0}(mn+m-1;q)} \\
&= \frac{1}{q + \frac{g_{m,0}(mn+m;q)}{g_{m,0}(mn+m-1;q)}} = \frac{1}{q + \frac{1}{\ell_m(n;q)}} \\
&= \ell_m(mn+m-1;q),
\end{aligned}$$

which implies that the claim holds for  $\ell_m(mn+m-1;q)$ . Thus, it remains to show that  $\frac{g_{m,0}(m(mn+m)+m-1;q)}{g_{m,0}(m(mn+m)+m;q)} = \ell_m(mn+m;q)$ . Let  $n_1 = n$  and  $n_j = mn_{j+1} + m$  for  $j = 1, 2, \dots, s-1$ , with  $s$  maximal. Thus,  $n_s = mn' + j$  for some  $j \in [m-1]$  or  $n_s = 0$ . By the definitions, we have

$$\begin{aligned}
\frac{g_{m,0}(m(mn+m)+m-1;q)}{g_{m,0}(m(mn+m)+m;q)} &= \frac{g_{m,0}(mn+m;q)}{g_{m,0}(mn+m+1;q) + qg_{m,0}(mn+m;q)} \\
&= \frac{1}{q + \frac{g_{m,0}(mn+m+1;q)}{g_{m,0}(mn+m;q)}} \\
&= \frac{1}{q + \ell_m(n;q) \frac{g_{m,0}(n+1;q)}{g_{m,0}(n;q)}},
\end{aligned}$$

where

$$\begin{aligned}
\frac{g_{m,0}(n_1+1;q)}{g_{m,0}(n_1;q)} &= \frac{g_{m,0}(mn_2+m+1;q)}{g_{m,0}(mn_2+m;q)} = \frac{g_{m,0}(n_2+1;q)}{g_{m,0}(n_2+1;q) + qg_{m,0}(n_2;q)} \\
&= \frac{1}{1 + q \frac{g_{m,0}(n_2;q)}{g_{m,0}(n_2+1;q)}} = \frac{1}{1 + q \frac{g_{m,0}(mn_3+m;q)}{g_{m,0}(mn_3+m+1;q)}} \\
&= \frac{1}{1 + q \frac{g_{m,0}(n_3+1;q) + qg_{m,0}(n_3;q)}{g_{m,0}(n_3+1;q)}} = \frac{1}{1 + q + q^2 \frac{g_{m,0}(n_3;q)}{g_{m,0}(n_3+1;q)}} \\
&= \dots \\
&= \frac{1}{1 + q + \dots + q^{s-2} + q^{s-1} \frac{g_{m,0}(n_s;q)}{g_{m,0}(n_s+1;q)}} \\
&= \frac{1}{1 + q + \dots + q^{s-2} + q^{s-1} \frac{g_{m,0}(mn'+j;q)}{g_{m,0}(mn'+j+1;q)}}.
\end{aligned}$$

Note that when  $j \in [m-2]$ , we have  $\frac{g_{m,0}(mn'+j;q)}{g_{m,0}(mn'+j+1;q)} = 1$ , and when  $j = m-1$ , we have  $\frac{g_{m,0}(mn'+j;q)}{g_{m,0}(mn'+j+1;q)} = \ell_m(n'; q)$ . It follows from the definitions that

$$\frac{g_{m,0}(m(mn+m)+m-1;q)}{g_{m,0}(m(mn+m)+m;q)} = \ell_m(mn+m;q),$$

which completes the induction.  $\square$

We have the following result when  $q = 1$  concerning the tree  $\mathcal{T}_m$ .

**Theorem 3.9.** *When  $q = 1$ , each positive rational number less than or equal one appears at least once in the  $(q, 0)$ -Calkin-Wilf tree of order  $m$  for all  $m \geq 2$ .*

*Proof.* We will show that all rational numbers  $\frac{a}{b}$  in lowest terms, where  $0 < \frac{a}{b} \leq 1$ , belong to  $\mathcal{T}_m$  when  $q = 1$  by inducting on the sum  $s = a + b$ , the case  $s = 2$  clear. First suppose  $0 < \frac{a}{b} \leq \frac{1}{2}$ . Then  $\frac{a}{b-a} \in \mathcal{T}_m$ , by hypothesis, and has  $(m-1)$ -st child  $\frac{a}{b}$ , which implies  $\frac{a}{b} \in \mathcal{T}_m$ .

So assume  $\frac{1}{2} < \frac{a}{b} < 1$ . We will construct a vertex whose label is  $\frac{a}{b}$ . To do so, let  $v$  be a vertex of  $\mathcal{T}_m$  labeled by  $\frac{x}{y}$ , where  $0 < \frac{x}{y} \leq \frac{1}{2}$  is in lowest terms and  $x$  and  $y$  are to be determined. Note that  $v$  can be taken to be an  $(m-1)$ -st child of a vertex labeled by  $\frac{x}{y-x}$ . Consider the sequence  $v = v_0, v_1, v_2, \dots$  of vertices of  $\mathcal{T}_m$  such that  $v_i$  is the  $m$ -th child of  $v_{i-1}$  for  $i \geq 1$ . Using the definitions, one can show by induction that the vertex  $v_i$  is labeled by  $\frac{iy-(i-1)x}{(i+1)y-ix}$  for all  $i \geq 0$ . Note that  $x$  and  $y$  relatively prime implies that each of these fractions is in lowest terms.

Suppose now that  $j \geq 1$  is determined by the condition  $\frac{j}{j+1} < \frac{a}{b} \leq \frac{j+1}{j+2}$ . Setting  $\frac{a}{b} = \frac{jy-(j-1)x}{(j+1)y-jx}$  implies  $x = (j+1)a - jb$  and  $y = ja - (j-1)b$ . Note that  $a$  and  $b$  relatively prime implies  $x$  and  $y$  are. Furthermore, using the restrictions on  $\frac{a}{b}$ , one can show that  $0 < \frac{x}{y} \leq \frac{1}{2}$ , as required. Finally, note that  $x + y = (2j+1)a - (2j-1)b < a + b$ , which implies  $\frac{x}{y} \in \mathcal{T}_m$ , by hypothesis. Thus, taking the  $m$ -th child exactly  $j$  times starting with any vertex labeled by  $\frac{x}{y}$  implies  $\frac{a}{b} \in \mathcal{T}_m$ , which completes the induction.  $\square$

### 3.3 Case $1 \leq c \leq m-2$

The remaining cases when  $1 \leq c \leq m-2$  may be described in terms of a single tree.

**Definition 3.10.** *Given  $m \geq 3$  and  $1 \leq c \leq m-2$ , the  $(q, c)$ -Calkin-Wilf tree of order  $m$  is an  $m$ -ary tree with root  $\frac{1}{1}$ . Each vertex labeled  $\frac{a}{b}$  is a parent to  $m$  children defined, from left to right, as follows: the  $k$ -th child for  $k \neq c, c+1$  is  $\frac{1}{1+q}$ , the  $c$ -th child is  $\frac{1}{1+q\frac{a}{b}}$ , and the  $(c+1)$ -st child is  $\frac{1}{1+q\frac{a}{b}}$ .*

**Theorem 3.11.** *Let  $m \geq 3$  and  $1 \leq c \leq m-2$ . Suppose that the concatenation of successive levels of the  $(q, c)$ -Calkin-Wilf tree of order  $m$  forms a sequence*

$\{\ell_{m,c}(n; q)\}_{n \geq 0}$ . Then

$$\ell_{m,c}(n; q) = \frac{g_{m,c}(mn + c - 1; q)}{g_{m,c}(mn + c; q)},$$

for all  $n \geq 0$ .

*Proof.* Let  $\mathcal{T}_{m,c}$  be the  $(q, c)$ -Calkin-Wilf tree of order  $m$ . We proceed by induction on  $n$ , the  $n = 0$  case clear. We again prove the claim for the children of  $\ell_{m,c}(n; q)$ . By the induction hypothesis and Definition 3.10, we have

$$\begin{aligned} \frac{g_{m,c}(m(mn + j) + c - 1; q)}{g_{m,c}(m(mn + j) + c; q)} &= \frac{g_{m,c}(mn + j; q)}{g_{m,c}(mn + j; q) + qg_{m,c}(mn + j - 1; q)} \\ &= \frac{g_{m,c}(n; q)}{g_{m,c}(n; q) + qg_{m,c}(n; q)} \\ &= \frac{1}{1 + q} = \ell_m(mn + j; q), \end{aligned}$$

for all  $j \in [m]$  and  $j \neq c, c + 1$ . When  $j = c$ , we have

$$\begin{aligned} \frac{g_{m,c}(m(mn + c) + c - 1; q)}{g_{m,c}(m(mn + c) + c; q)} &= \frac{g_{m,c}(mn + c; q)}{g_{m,c}(mn + c; q) + qg_{m,c}(mn + c - 1; q)} \\ &= \frac{1}{1 + q \frac{g_{m,c}(mn + c - 1; q)}{g_{m,c}(mn + c; q)}} \\ &= \frac{1}{1 + q\ell_{m,c}(n; q)} \\ &= \ell_{m,c}(mn + c; q), \end{aligned}$$

which implies that the claim holds for  $\ell_{m,c}(mn + c; q)$ . Finally, when  $j = c + 1$ , we have

$$\begin{aligned} \frac{g_{m,c}(m(mn + c + 1) + c - 1; q)}{g_{m,c}(m(mn + c + 1) + c; q)} &= \frac{g_{m,c}(mn + c + 1; q)}{g_{m,c}(mn + c + 1; q) + qg_{m,c}(mn + c; q)} \\ &= \frac{1}{1 + q \frac{g_{m,c}(mn + c; q)}{g_{m,c}(mn + c + 1; q)}} \\ &= \frac{1}{1 + q \frac{g_{m,c}(mn + c; q)}{g_{m,c}(mn + c - 1; q)}} \\ &= \frac{1}{1 + q \frac{1}{\ell_{m,c}(n; q)}} \\ &= \ell_{m,c}(mn + c + 1; q), \end{aligned}$$

which implies that the claim holds for  $\ell_{m,c}(mn + c + 1; q)$  and completes the induction.  $\square$

*Remark:* When  $q = 1$ , each positive rational number less than or equal 1 appears at least once in  $\mathcal{T}_{m,c}$  as a fraction in lowest terms for all  $m \geq 3$  and  $1 \leq c \leq m - 2$ .

This follows from the definitions, upon inducting on the sum  $a + b$  corresponding to a vertex labeled by the fraction  $\frac{a}{b}$  in lowest terms. Indeed, each rational in the interval  $(0, 1)$  is seen to occur infinitely many times in the tree since it essentially starts over each time a vertex is labeled by  $\frac{1}{2}$ .

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